Lotka-Volterra Equation over a Finite Ring $\mathbf{Z}/p^N\mathbf{Z}$

Shigeki Matsutani
8-21-1 Higashi-Linkan Sagamihara 228-0811 JAPAN

Abstract

Discrete Lotka-Volterra equation over p-adic space was constructed since p-adic space is a prototype of spaces with the non-Archimedean valuations and the space given by taking ultra-discrete limit studied in soliton theory should be regarded as a space with the non-Archimedean valuations in the previous report (solv-int/9906011). In this article, using the natural projection from p-adic integer to a ring $\mathbf{Z}/p^n\mathbf{Z}$, a soliton equation is defined over the ring. Numerical computations shows that it behaves regularly.

§1. Introduction

According to [LL], studies on the ultrametric space which is characterized by the strong triangle axiom

$$d(x,z) \leq \max[d(x,y), d(y,z)] \tag{1}$$

or for one-dimensional additive space case

$$|x-z|_{\rm ult} \leq \max[|x|_{\rm ult}, |z|_{\rm ult}] \tag{2}$$

has been current for this 10-15 years in the fields of general topology, computer language, rings of meromorphic function and so on. This space is called non-Archimedean space in English and German literature and is known as ultrametric space in France literature and as isosceles space in Russian [LL]. This space appeared, in first, in the theory of number theory as p-adic (Hensel) integer but nowadays, it is known that this space is natural even for the fields out of number theory.

In fact, the space obtained by ultra-discrete limit which has been studied in soliton theory [TS, TTMS] holds the relation (1) as shown in [M]. (If one recognizes that the soliton theory is roughly a theory of functions over a compact Riemann surface, the functions are also governed by the non-Archimedean valuation [I] and thus the ultrametric is built in soliton theory. Sato theory is based on the fact [S].)

In the studies of the ultrametric space, it is natural to feel that the theory over a field with characteristics $q = 0$ is too restricted because p-adic space is a prototype of the ultrametric space.

Actually as the ultra-discrete equation [TS, TTMS] is given as a difference-difference equation, difference-difference equations are in general given by algebraic relations. Thus it is natural to consider the equations in the algebraic category and there non-vanishing characteristics is canonical. For example, the one-dimensional linear difference-difference heat equation is algebraically defined as follows: Let K be a field with characteristics $q = 0$ and $K[X,T]$ be a set of polynomial of X and T. Let us introduce a ring $F_X[T] := K[X,T]/(1-X^n)$ where $(1-X^n)$ is an ideal generated by $1-X^n$ and n is a positive integer. We have its subset

$$F_X^i[T] := \{ f \in F_X[T] \mid \text{order of } f \text{ in } T \leq i \ \}. \tag{3}$$

Then we can define a map $\phi : F_X^i[T]/F_X^{i-1}[T] \to F_X^{i+1}[T]/F_X^i[T]$ by for an element f_i of $F^i[X,T]/F^{i-1}[X,T]$, $\phi f_i := T(X-2+\epsilon+X^{-1})f_i$. For an element $f_0 \in F_X^0[X,T]$, we compute $f_n := \phi^n f_0$. This is the computation of the one-dimensional discrete heat equation with a periodic boundary condition on n; X and T are shift operators. Since $F_X^0[T]$ can be regarded as a cyclotomic field, one may compare the norm of $|X-2+\epsilon+X^{-1}|$ with the parameter ϵ, which is related to stability's criterion of the numerical computation [PTVF]. In this formulation, it is not strange to consider it over a field with non-vanishing characteristics.

Similarly we wish to formulate the difference-difference non-linear equation over a more general field with non-vanishing characteristics or its related ring. In fact, there were several attempts to formulate the soliton theory over finite fields [N, NM]. The purpose of this study including the previous report [M] is to extend the soliton theory over $\mathbf{R}$ to p-adic number space in order to consider the meanings of

the ultra-discrete limit. (As the p-adic valuation is a prototype of the non-Archimedean valuation and the ultra-discrete system is natural in soliton theory, it is very natural to investigate soliton equations in the p-adic space.) The extension of soliton theory to p-adic space has been done by Ichikawa for continuous soliton theory or KP-hierarchy [Ic1-3]. In this study, we restrict ourselves only to consider the difference-difference equations. Then as mentioned in [M], we can also define the p-adic difference-difference Lotka-Volterra equation and show that its p-adic valuation version has the same structure of the ultra-discrete difference-difference Lotka-Volterra equation. This implies that there might exist a functor between categories of discrete ordinary soliton equations and p-adic soliton equations, even though we did not mention it [M].

In this article, we will investigate p-adic system more concretely. Since p-adic integer is canonically connected with a finite ring $\mathbf{Z}/p^N\mathbf{Z}$, we will consider that the Lotka-Volterra equation over the finite ring $\mathbf{Z}/p^N\mathbf{Z}$. Due to its finiteness, we can compute it concretely. Numerical computations show that the Lotka-Volterra equation over the finite ring $\mathbf{Z}/p^n\mathbf{Z}$ behaves regularly. It is expected that there is a natural group governing the system.

In this article, we denote the set of integers, rational number and real numbers by $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ respectively.

§2. Ultra-Discrete Limit as a Valuation

This section reviews the previous report [M] briefly in order to connect the ultra-discrete system with ultrametric system. Let $\overline{\mathcal{A}_{[\beta]}}$ be a set of non-negative real valued functions over $\{\beta \in \mathbf{R}_{>0}\}$ where $\mathbf{R}_{>0}$ is a set of positive real numbers. Let us define a map $\mathrm{ord}_\beta : \overline{\mathcal{A}_{[\beta]}} \cup \{0\} \to \mathbf{R} + \infty$. We set $\mathrm{ord}_\beta(0) = \infty$ for $u \equiv 0$ and for $u \in \overline{\mathcal{A}_{[\beta]}}$,

$$\mathrm{ord}_\beta(u) := -\lim_{\beta\to+\infty} \frac{1}{\beta}\log(u). \tag{4}$$

We call this value ultra-discrete of u.

Let us choose a subset $\mathcal{A}_{[\beta]}$ of $\overline{\mathcal{A}_{[\beta]}}$, $\mathcal{A}_{[\beta]} := \{u \in \overline{\mathcal{A}_{[\beta]}} \mid \mathrm{ord}_\beta(u) < \infty \}$. Further we identify the set $\{u \in \overline{\mathcal{A}_{[\beta]}} \mid \mathrm{ord}_\beta(u) = \infty \}$ with $\{0\}$. The ultra-discrete ord_β is a non-Archimedean valuation [I] since it hold the properties (I_β):

Proposition $\mathbf{I}_\beta$ *For $u, v \in \mathcal{A}_{[\beta]} \cup \{0\}$,*

1. $\mathrm{ord}_\beta(uv) = \mathrm{ord}_\beta(u) + \mathrm{ord}_\beta(v)$.
2. $\mathrm{ord}_\beta(u+v) = \min(\mathrm{ord}_\beta(u), \mathrm{ord}_\beta(v))$.

Let us, now, give the difference-difference Lotka-Volterra equation for $\{c_n^m \in \mathbf{R}_{\geq 0} \mid (n,m) \in \Omega \times \mathbf{Z} \}$ [HT],

$$\frac{c_n^{m+1}}{c_n^m} = \frac{1+\delta c_{n-1}^m}{1+\delta c_{n+1}^{m+1}}. \tag{5}$$

Here Ω is a subset of $\mathbf{Z}$, δ is a small parameter ($|\delta| < 1$) connecting between discrete system and continuum system, n is an index of a subset of the integer $\mathbf{Z}$ and m is of time step.

By introducing new variables $f_n^m := -\mathrm{ord}_\beta(c_n^m)$ and $d := -\mathrm{ord}_\beta(\delta)$ [T], we have a ultra-discrete version of the difference-difference Lotka-Volterra equation (5) for $c_n^m \in \mathcal{A}_{[\beta]}$ and $\delta \in \mathcal{A}_{[\beta]}$ [TS, T, TTMS],

$$f_n^{m+1} - f_n^m = \mathrm{ord}_\beta(1+\delta_p c_{n-1}^m) - \mathrm{ord}_\beta(1+\delta_p c_{n-1}^m). \tag{6}$$

We emphasize that (6) is considered as a valuation version of the difference-difference soliton equation (5).

Now in order to connect the ultra-discrete valuation and ultrametric in the framework [LL], we introduce a real number $\overline{\beta} \gg 1$ and define a quantity for $x \in \mathcal{A}_{[\beta]}$ as,

$$|x|_\beta := \left(\mathrm{e}^{-\bar\beta}\right)^{\mathrm{ord}_\beta(x)}. \tag{7}$$

This is an ultrametric because it satisfies next proposition.

Proposition II$_\beta$ *For u, $v \in \mathcal{A}_{[\beta]} \cup \{0\}$, $|u|_\beta$ and $|v|_\beta$ hold following properties,*

1. *$|u|_\beta$ depends upon $\bar\beta$.*
2. *if $|v|_\beta = 0$, $v = 0$.*
3. *$|v|_\beta \ge 0$.*
4. *$|vu|_\beta = |v|_\beta |u|_\beta$.*
5. *$|u + v|_\beta = \max(|u|_\beta, |v|_\beta) \le |u|_\beta + |v|_\beta$.*

Here we will remark on this ultrametric as follows [M].

1. If we define the distance $d(x, y)$ between points $x, y \in \mathcal{A}_{[\beta]} \cup \{0\}$ by $d(x, y) := |\ |u - v|\ |_\beta$, the fifth property in II$_\beta$ satisfies (2), since the absolute value $|u - v|$ belongs to $\mathcal{A}_{[\beta]} \cup \{0\}$. This metric induces very a week topology.
2. Since $x \in \mathcal{A}_{[\beta]}$ has a finite value at $\beta \to \infty$, we have relation
$$|x|_\beta|_{\bar\beta\sim\infty} \sim \exp(-\bar\beta(-(\log x)/\beta))\big|_{\bar\beta\sim\beta\sim\infty} = |x|^{\bar\beta/\beta}|_{\bar\beta\sim\beta\sim\infty}. \tag{8}$$
It may be regarded that $|x|_\beta \sim |x|$, in heart, by synchronizing $\bar\beta$ and β. $|x|_\beta$ is consist with the natural metric $|\cdot|$ in $\mathbf{R}$.
3. In this metric, we have the relation,
$$|\sum_m x_m|_\beta = \mathrm{e}^{-\bar\beta \min(\mathrm{ord}_\beta(x_m))}. \tag{9}$$
This relation appears in the partition function at $\bar\beta \sim \beta = 1/T$, $T \to 0$ and in the semi-classical path integral $\bar\beta \sim \beta = 1/\hbar$, $\hbar \to 0$ [D, FH]. It means that *the classical regime appears as a non-Archimedean valuation, which is an algebraic manipulation.* For example, a problem with a minimal principle might be regarded as a valuation of a quantum problem.

These remarks shows that the ultrametric obtained from the ultra-discrete is a very natural object from physical and mathematical viewpoints.

§3. Preliminary: p-adic Space

In the ultrametric space theory, p-adic space is prototype. Hereafter let us consider the p-adic space. In this section, let us introduce p-adic field $\mathbf{Q}_p$ and p-adic integer $\mathbf{Z}_p$ for a prime number p [C, I, L, RTV, VVZ]. For a rational number $u \in \mathbf{Q}$ which is given by $u = \dfrac{v}{w}p^m$ (v and w are coprime to the prime number p and m is an integer), we define a symbol $[[u]]_p = p^m$. Here let us define the p-adic valuation of u as a map from $\mathbf{Q}$ to a set of integers $\mathbf{Z}$,

$$\mathrm{ord_p}(u) := \log_p[[u]]_p, \quad \text{for } u \neq 0, \quad \text{and} \quad \mathrm{ord_p}(u) := \infty, \quad \text{for } u = 0. \tag{10}$$

This valuation has following properties (I$_p$), which is similar to I$_\beta$ of ord$_\beta$,

Proposition I$_p$: *For $u, v \in \mathbf{Q}$,*

1. *$\mathrm{ord_p}(uv) = \mathrm{ord_p}(u) + \mathrm{ord_p}(v)$.*

2. $\mathrm{ord_p}(u+v) \geq \min(\mathrm{ord_p}(u), \mathrm{ord_p}(v))$.
   If $\mathrm{ord_p}(u) \neq \mathrm{ord_p}(v)$, $\mathrm{ord_p}(u+v) = \min(\mathrm{ord_p}(u), \mathrm{ord_p}(v))$.

This property (I_p-1) means that $\mathrm{ord_p}$ is a homomorphism from the multiplicative group $\mathbf{Q}^\times$ of $\mathbf{Q}$ to the additive group $\mathbf{Z}$. The p-adic metric is given by $|v|_p = p^{-\mathrm{ord_p}(v)}$, which has the properties (II_p);

Proposition II_p: *For* $u, v \in \mathbf{Q}$,

1. *if* $|v|_p = 0$, $v = 0$.
2. $|v|_p \geq 0$.
3. $|vu|_p = |v|_p |u|_p$.
4. $|u+v|_p \leq \max(|u|_p, |v|_p) \leq |u|_p + |v|_p$.

The p-adic field $\mathbf{Q}_p$ is given as a completion of $\mathbf{Q}$ with respect to this metric so that properties (I_p) and (II_p) survive for $\mathbf{Q}_p$.

It should be noted that the properties I_p and II_p are essentially the same as those of I_β and II_β. As a property of p-adic metric, the convergence condition of series $\sum_m x_m$ is identified with the vanishing condition of sequence $|x_m|_p \to 0$ for $m \to \infty$ due to the relationship [C, L, VVZ],

$$|\sum_m x_m|_p = \max |x_m|_p. \tag{11}$$

This property is related to (9).

Further we note that the integer part of $\mathbf{Q}_p$, $\mathbf{Z}_p := \{u \in \mathbf{Q}_p \mid \mathrm{ord_p}(u) > 0\}$, is a *localized ring* and has only prime ideals $\{0\}$ and $p\mathbf{Z}_p$ [L]. $\mathbf{Z}_p$ can be also defined by an inverse limit of the projective sequence [L],

$$\mathbf{Z}/p\mathbf{Z} \leftarrow \mathbf{Z}/p^2\mathbf{Z} \leftarrow \mathbf{Z}/p^3\mathbf{Z} \leftarrow \cdots, \tag{12}$$

or

$$\mathbf{Z}_p := \lim_{\leftarrow} \mathbf{Z}/p^N\mathbf{Z}. \tag{13}$$

Thus there is a natural surjective ring homomorphism from $\mathbf{Z}_p$ to $\mathbf{Z}/p^N\mathbf{Z}$.

§4. p-adic Difference-Difference Lotka-Volterra Equation and Its Applications

Now let us define the p-adic difference-difference Lotka-Volterra equation for a p-adic series $\{c_n^m \in \mathbf{Q}_p \mid (n, m) \in \Omega \times \mathbf{Z}\}$ ($p \neq 2$),

$$\frac{c_n^{m+1}}{c_n^m} = \frac{1 + \delta_p c_{n-1}^m}{1 + \delta_p c_{n+1}^{m+1}}, \tag{14}$$

or

$$c_n^{m+1}(1 + \delta_p c_{n+1}^{m+1}) = c_n^m(1 + \delta_p c_{n-1}^m), \tag{15}$$

where $\delta_p \in p\mathbf{Z}_p$ and $|\delta_p|_p < 1$. We proved that this equation has non-trivial solution in [M].

In this article, we will give another proof. Due to (12) and (13), there is a natural projection from $\mathbf{Z}_p$ to $\mathbf{Z}/p^N\mathbf{Z}$. The equation (15) can be solved by module computations if $c_n^0 \in \mathbf{Z}_p$ for all $n \in \Omega$. Let us expand c_n^m in p-adic space,

$$c_n^m := \alpha_n^{m(0)} + \alpha_n^{m(1)} p + \alpha_n^{m(2)} p^2 + \cdots, \quad c_n^{m(N)} := \sum_{i=0}^{N-1} \alpha_n^{m(i)} p^i. \tag{16}$$

For simplicity, we assume $\delta_p \equiv p$. By comparing the coefficients of p^N $(N = 0, 1, 2, \cdots)$ in the both sides in (15), we can determine the time revolution iteratively:

$$
\begin{aligned}
\alpha_n^{m+1\,(0)} &= \alpha_n^{m\,(0)}, \\
\alpha_n^{m+1\,(1)} &= \left(c_n^m(1+pc_{n-1}^m) - c_n^{m+1\,(1)}\right)/p \quad \text{module } p, \\
&\cdots \\
\alpha_n^{m+1\,(N)} &= \left(c_n^m(1+pc_{n-1}^m) - c_n^{m+1\,(N)}(1+p\;c_{n+1}^{m+1\,(N)})\right)/p^N \quad \text{module } p.
\end{aligned} \tag{17}
$$

This comparison means that we compute (15) in modulo p^{N+1}, $(N = 0, 1, 2, \cdots)$. If the initial state is given by $\{c_n^0 \equiv c_n^{0\,(N_0)}\}_{n\in\Omega}$ for a finite N_0, above computations give all values of $\alpha_n^{m\,(N)}$'s. Then $\alpha_n^{m\,(N)}$'s vanish for $n \in \Omega$, $N > N_1$, sufficient large N_1, and finite m. It implies that (15) has non-trivial solutions in p-adic space.

Using the natural projection from $\mathbf{Z}_p$ to $\mathbf{Z}/p^N\mathbf{Z}$, we have solutions (17) of the equation (15) modulo p^{N+1} when we fix N. We give some examples in tables 1, 2, and 3 with periodic boundary condition for n; $c_n^m \equiv c_{n+M}^m$ and $M = 5$. These are of $p = 3$ and $p = 5$ modulo p^3 $(N = 2)$ cases and $p = 7$ modulo p^2 $(N = 1)$ case. Numerical computations show that they are also periodic on time m $c_n^m = c_n^{m+p^N}$. Since $\alpha_n^{m\,(0)}$ is an invariance, the excitations look localized. We can find that there appear several symmetries; in the tables c_4^m oscillate with shorter periods p^{N-1}, c_n^m has point symmetry centerizing at $(n = 1.5, m = p^N/2)$ and so on. We regard that these are solutions of Lotka-Volterra over rings $\mathbf{Z}/p^N\mathbf{Z}$. It is also noted that (15) over the finite field $\mathbf{F}_p \equiv \mathbf{Z}/p\mathbf{Z}$ gives only trivial solutions since $\alpha_n^{m+1\,(0)}$ is invariant. These facts mean that *we can define soliton equation over rings beside finite fields* [N, NM].

Here we will mention the relation of p-adic equation (15) to the ultra-discrete system following [M]. As the p-adic difference-difference Lotka-Volterra equation is well-defined, let us consider the p-adic valuation of the equation (14) even though the conserved quantities $\alpha_n^{m+1\,(0)}$ might make the valuation trivial. By letting $f_n^m := -\mathrm{ord}_\mathrm{p}(c_n^m)$ and $d_p := -\mathrm{ord}_\mathrm{p}(\delta_p)$, we have

$$f_n^{m+1} - f_n^m = \mathrm{ord}_\mathrm{p}(1 + \delta_p c_{n-1}^m) - \mathrm{ord}_\mathrm{p}(1 + \delta_p c_{n-1}^m). \tag{18}$$

For $f_n^m \neq -d_p$, (18) becomes

$$f_n^{m+1} - f_n^m = \max(0, f_{n-1}^m + d_p) - \max(0, f_{n+1}^{m+1} + d_p). \tag{19}$$

We emphasize that (19) has the same form as the ultra-discrete difference-difference Lotka-Volterra equation (6) [M]. We should note that p-adic valuation is a natural object in the p-adic number and the family of rings $\{\mathbf{Z}/p^N\mathbf{Z}\}$. This implies that the ultra-discrete difference-difference system should be also studied from the point of view of valuation theory [M].

As we finish this section, we will give a comment on a relation to q-analysis. It is known that some of properties in the q analysis can be regarded as those in p-adic analysis by setting $q = 1/p$ [VVZ]. We have correspondence among p, q and e^β as [M],

$$\mathrm{e}^{-\beta} \Longleftrightarrow p \;(\beta \sim \infty), \quad p \Longleftrightarrow 1/q, \quad q \Longleftrightarrow \mathrm{e}^\beta \;(\beta \sim 0). \tag{20}$$

§5. Summary and Discussion

We showed that the ultra-discrete limit should be regarded as a non-Archimedean valuation following the previous report [M]. After we constructed the ultrametric related to the ultra-discrete limit in (7), we remarked its properties. Due to the remarks, it is interpreted that the ultra-discrete limit is a very natural manipulation in $\mathcal{A}_\beta \cup \{0\}$.

Generally in the studies of the ultrametric space [LL], the p-adic system is a prototype. Thus we have considered p-adic soliton equation following the previous report [M]. In fact the structure of p-adic valuation of the p-adic difference-difference equation has the same structure as the ultra-discrete difference-difference equation for the case of Lotka-Volterra equation.

Further since p-adic field has a natural projection to a finite ring $\mathbf{Z}/p^N\mathbf{Z}$, we have studied the Lotka-Volterra equation over the finite ring. Due to the finiteness of system, we can give concrete solutions of the equation. There remains a problem what is integrability in the sense of $\mathbf{Z}/p^N\mathbf{Z}$ but the numerical computations give regular results and beautiful symmetries of the system. It is expected that there is a group governing this system. This construction can be easily extended to a p-adic system related to more general algebraic integer if the algebraic integer is a principal domain. As the soliton theory in finite field is closely related to the code theory [N, NM], the soliton over $\mathbf{Z}/p^N\mathbf{Z}$ might be also applied to the information theory [K]. Further as the discrete heat equation can be described algebraically in the introduction, we wish, in future, to express the equation over $\mathbf{Z}/p^N\mathbf{Z}$ more algebraically.

Finally we comment upon an open problem. The non-Archimedean valuation theory is associated with the measure theory or non-standard statistics [LL] and renormalization theory [RTV]. On the other hand, soliton theory is connected with statistical system and statistical mechanics [So]. Thus we have a question whether both non-standard statistics and soliton theory have more directly relation.

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Reference

[C] J.W.S. Cassels, *Lectures on Elliptic Curves* , (1991, Cambridge Univ. Press, Cambridge).

[D] P. A. M. Dirac, *The Principles of Quantum Mechanics, forth edition*, (1958, Oxford, Oxford).

[FH] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integral*, (1965, McGraw-Hill, Auckland).

[HT] R. Hirota and S. Tsujimoto, *Conserved Quantities of a Class of Nonliear Difference-Difference Equations* , J. Phys. Soc. Jpn.**64** (1995) 3125-3127.

[Ic1] T. Ichikawa, *The universal periods of curves and the Schottky problem* , Comp. Math. **85** (1993) 1-8.

[Ic2] —-, *p-adic theta functions and solutions of the KP hierarchy* , Comm. Math. Phys. **176** (1996) 383-399.

[Ic3] —-, *Schottky uniformarization theory on Riemann surfaces and Mumford curves of infinite genus* J. reine angew. Math. **486** (1997) 45-68.

[I] K. Iwasawa, *Algebraic Function Theory*, (1952, Iwanami, Tokyo) *japanese*.

[K] N. Koblitz, *Algebraic Aspects of Cryptography* (1998, Springer, Berlin).

[L] K. Lamotke (ed.), *Zahlen* (1983, Springer, Berlin).

[LL] A. J. Lemin and V. A. Lemin, *On a universal ultrametric space* , Topology and its Applications **103** (2000) 339-345.

[M] S. Matsutani, *p-adic Difference-Difference Lotka-Volterra Equation and Ultra-Discrete Limit*, Int. J. Math. and Math. Sci. (2001) to appear, (solv-int/9906011).

[N] Y. Nakamura, *The BCH-Goppa decoding as a moment problem and a tau function over finite fields* , Phys. Lett. A **223** (1996) 75-81.

[NM] Y. Nakamura and A. Mukaihira, *Dynamics of the finite Toda molecule over finite fields and a decoding algorithm* , Phys. Lett. A **249** (1998) 295-302.

[RTV] R. Rammal, G. Toulouse and M. A. Virasoro, *Ultrametricity for physicists* , Rev. Mod. Phys.**58** (1986) 765-788.

[PTVF] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery *Numerical Recipes in Fortran 77: The Art of Scientific Computing (Vol. 1 of Fortran Numerical Recipes), 2nd ed.*, (1996, Cambridge Univ. Press, Cambridge).

[S] M. Sato, *Soliton Equation and Universal Grassmannian Manifold* , Lecture note, noted by M. Noumi, (*japanese*) (1984, Sophia Univ, Tokyo).

[So] K. Sogo, *Time-Dependent Orthogonal Polynomials and Theory of Soliton - Applications to Matrix Model, Vertex Model and Level Statistics* , J. Phys. Soc. Jpn **62** (1993) 1887-1894.

[T] D. Takahashi, *Ultra-discrete Toda Lattice Equation -A Grandchild of Toda-* , International Symposium, Advances in soliton theory and its applications: The 30th anniversary of the Toda lattice (1996) 36-37.

[TS] D. Takahashi and J. Satsuma, *A Soliton Cellular Automaton* , J. Phys. Soc. Jpn.**59** (1990) 3514-3519

[TTMS] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, *From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure* , Phys.Rev.Lett. **76** (1996) 3427-3250.

[VVZ] V.S.Vladimirov, I.V.Volvich and E.I.Zelenov, *P-adic Analysis and Mathematical Physics* (1994, World Scientific, Singapore).

Table 1: modulo 3^3, $\delta_p = 3$

$m \setminus n$	0	1	2	3	4	5
0	1	2	2	1	1	1
1	7	23	26	22	10	7
2	13	8	14	16	10	13
3	19	11	20	10	1	19
4	25	5	17	4	10	25
5	4	17	5	25	10	4
6	10	20	11	19	1	10
7	16	14	8	13	10	16
8	22	26	23	7	10	22
9	1	2	2	1	1	1

Table 2: modulo 5^3, $\delta_p = 5$

$m \setminus n$	0	1	2	3	4	5
0	1	2	2	1	1	1
1	96	117	37	106	26	96
2	16	7	97	36	101	16
3	11	47	57	41	101	11
4	81	112	42	121	26	81
5	101	77	52	26	1	101
6	71	67	87	6	26	71
7	116	82	22	61	101	116
8	111	122	107	66	101	111
9	56	62	92	21	26	56
10	76	27	102	51	1	76
11	46	17	12	31	26	46
12	91	32	72	86	101	91
13	86	72	32	91	101	86
14	31	12	17	46	26	31
15	51	102	27	76	1	51
16	21	92	62	56	26	21
17	66	107	122	111	101	66
18	61	22	82	116	101	61
19	6	87	67	71	26	6
20	26	52	77	101	1	26
21	121	42	112	81	26	121
22	41	57	47	11	101	41
23	36	97	7	16	101	36
24	106	37	117	96	26	106
25	1	2	2	1	1	1

Table 3: modulo 7^2, $\delta_p = 7$

$m \setminus n$	0	1	2	3	4	5
0	1	2	2	1	1	1
1	43	37	16	8	1	43
2	36	23	30	15	1	36
3	29	9	44	22	1	29
4	22	44	9	29	1	22
5	15	30	23	36	1	15
6	8	16	37	43	1	8
7	1	2	2	1	1	1